'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 1st Semester Examination, 2021

CC2-MATHEMATICS

ALGEBRA

Time Allotted: 2 Hours　　　　　　　　　　　　　　　　Full Marks: 60

The figures in the margin indicate full marks.
All symbols are of usual significance.

GROUP-A

1. Answer any ***four*** questions: $3\times 4 = 12$

(a) Find the remainder when $1! + 2! + 3! + \cdots\cdots + 50!$ is divided by 15.

(b) If one of the roots of the equation $x^3 + px^2 + qx + r = 0$ is equal to the sum of the other two roots, prove that $p^3 + 8r = 4pq$.

(c) Find the value of $\sum \frac{1}{\beta+\gamma}$, if α, β, γ be the roots of the equation $x^3 + px + q = 0$.

(d) Prove that $(n+1)^n > 2^n \cdot n!$.

(e) Find the minimum number of complex roots of the equation $x^7 - 3x^3 + x^2 = 0$. (Descart's rule of sign may be applied).

(f) If A and P are $n\times n$ matrices and P is non-singular, then show that A and $P^{-1}AP$ have the same eigenvalues.

GROUP-B

Answer any *four* questions $6\times 4 = 24$

2. Show that $\frac{n^{11}}{11} + \frac{n^3}{3} + \frac{19n}{33}$ is integer for every natural number n.

3. Solve the following biquadratic equation by Ferrari's method

$$2x^4 + 5x^3 - 8x^2 - 17x - 6 = 0$$

4. Verify Cayley-Hamilton theorem for the matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Hence find A^{-1} and A^9.

5. If the roots of $x^3 + 2x^2 + 3x + 1 = 0$ are α, β, γ then find the equation whose roots are $\frac{1}{\alpha^2} + \frac{1}{\beta^2} - \frac{1}{\gamma^2}$, $\frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{1}{\alpha^2}$ and $\frac{1}{\gamma^2} + \frac{1}{\alpha^2} - \frac{1}{\beta^2}$.

6. Find the real value of λ such that the system has a non-zero solution:

$$x + 2y + 3z = \lambda x$$
$$3x + y + 2z = \lambda y$$
$$2x + 3y + z = \lambda z$$

7. If $a_1, a_2, \cdots\cdots, a_n$ and $t_1, t_2, \cdots\cdots, t_n$ be two lists of real numbers then show that $(a_1t_1 + a_2t_2 + \cdots\cdots + a_nt_n)^2 \le (a_1^2 + a_2^2 + \cdots\cdots + a_n^2)(t_1^2 + t_2^2 + \cdots\cdots + t_n^2)$ and equality holds when $\frac{a_1}{t_1} = \frac{a_2}{t_2} = \cdots\cdots = \frac{a_n}{t_n}$.

GROUP-C

Answer any *two* questions 12×2=24

8. (a) Prove that $8abc < (1-a)(1-b)(1-c) < \frac{8}{27}$, if a, b, c are not same and $a + b + c = 1$. 5

(b) If p is a prime and a, b are positive integers, show that 3

$$(a+b)^p \equiv (a+b) \pmod{p}$$

(c) Find the value of $(-i)^{2/5}$. 4

9. (a) If $x_1 > x_2 > x_3 > \cdots\cdots > x_n$, prove that 5

$$(x_1 - x_2)(x_2 - x_3)\cdots\cdots(x_{n-1} - x_n) < \left(\frac{x_1 - x_n}{n-1}\right)^{n-1}$$

(b) Find a non-singular matrix P such that $P^T AP$ is the normal form of A under congruence, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. 4+3

Hence find rank, index and signature of A.

10.(a) If a mapping $f : (-1, 1) \longrightarrow \mathbb{R}$ is defined by $f(x) = \frac{x}{1-|x|}$ $\forall\, x \in (-1, 1)$, then show that it is a continuous bijection. Find its inverse and check continuity of this inverse. 2+3+1+2

(b) Reduce the matrix A to row reduced echelon form and hence find its rank where 4

$$A = \begin{pmatrix} 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}$$

11.(a) If $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ and if p is prime to n, prove that 6

$$1 + \alpha^p + \alpha^{2p} + \cdots\cdots + \alpha^{(n-1)p} = 0$$

(b) If $a \equiv b \pmod{m}$ then show that $a^n \equiv b^n \pmod{m}$ for all possible integers n. Show by example that the converse of the result is not hold in general. 4+2

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